International Journal of Theoretical Physics, Vol. 42, No. 12, December 2003 (© 2003)

Discrete Time and Intrinsic Length in Quantum Mechanics

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Received March 9, 2003

We consider the possibility that simultaneously time and intrinsic length can be regarded as discrete real parameters. We study the dynamics of the free particle. For both scattering and bound states there are configurations where the energy is bounded from above and from below even for positive wave-function solutions. For the case of continuous evolution we show that the wave equation with a linear scalar coupling describes an oscillator that has built-in hidden supersymmetry.

KEY WORDS: discrete time; intrinsic length; quantum mechanics.

Throughout the development of quantum mechanics, time traditionally appears as a continuous parameter. In Feynman's path integration formulation for a nonrelativistic particle, the probability amplitude for the particle to be at an initial position at a time $t = t_{\mathrm{i}}$ and at a final position at $t = t_{\mathrm{f}}$ is given by the amplitude sum over all paths connecting the initial and final positions, apart from a normalization constant. It is clear that the position $\mathbf{x}$ of the particle is not treated on the same basis as its (real) time t: at a given time the path integration can be seen as over the whole range of eigenvalues of the position operator. This thus points out the familiar difference between $\mathbf{x}$ as an *operator* and t as a *parameter*.

This asymmetry is also clear in classical mechanics. The classical trajectory of a particle is determined by the extremity of the action, which is functional to the position. While $\mathbf{x}$ is the dynamical variable, t appears only as a continuous parameter. By setting the variational derivative, we obtain the usual Lagrange equation of motion, whose solution gives the classical path.

Our interest here is in the construction of a quantum mechanics with well-specified equations of motion discrete time and discrete *intrinsic* space. This is motivated by the notion that at some small scale, time and space would be really discrete. This has echoes in theories such as relativistic quantum mechanics with a time associated with the electron's Compton wavelength, and string theory, where

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0020-7748/03/1200-2831/0 © 2003 Plenum Publishing Corporation

the Planck time sets a scale at which conventional notions of space and time break down.

In particle physics, a widely adopted convention is to use as the fundamental units certain physical quantities which are *constants of nature*. Two constants of this sort are c, the speed of light in vacuum, and $\hbar$, Planck's bar constant. We usually choose these constants as two of the fundamental units of our systems. As a third unit we use the second or the centimeter as the conventional and arbitrary unit of time or length. This choice is guided by the fact that the theory under discussion arises from an intimate relation between special relativity characterized by the constant c and quantum mechanics characterized by $\hbar$. The need for a theory involving fundamental time and length units has been the subject of much speculation in the past and present but it seems safe to say that we are far from understanding the role of such units in existing theories.

There are several situations in physics where it is convenient or necessary to replace the continuous time (temporal evolution) and the continuous length parameters with a discrete parameters. There have been various attempts to construct classical and quantum mechanical theories on the basis of these notions, such as the works of Lee (1983), Caldirola (1978), Kadyshevsky (1978; Kadyshevsky and Mateev, 1981), Golden (1991), and Farias and Recami (1997). The works of Yamamoto *et al.* (1995), Hashimoto *et al.* (1995), Klimek (1993), Jaroszkiewicz and Norton (1997), Milburn (1991, 1998), and Bruce (2001) show that the subject continues to receive attention.

The underlying postulate is that on sufficiently short-time steps the system does not develop continuously under unitary evolution but rather in a sequence of *identical* transformations. The inverse of this time step is the mean frequency of the steps. If the time step and consequently the space step are small enough, the evolution appears approximately continuous on laboratory time scales. To zeroth order the Schrödinger equation is recovered.

Often we utilize l, m, s (mks or cgs), or rather $c, \hbar, s$ (natural units), as basic unit systems. Here we shall consider a different set, namely either $(c, \hbar, \tau)$ or $(c, \hbar, \lambda)$ as unit systems, where τ and λ are natural units of time and length, respectively.

Given the fact that time and space appear in the free particle Dirac operator as derivatives with respect to time and space, we conjecture that a scalar term also enters as a single partial derivative in the mass term of the Dirac operator. Notice that, like the time derivative on the Dirac states Ψ, this term is always present in the Dirac hamiltonian, even in the rest mass frame system of the particle. Thus for *continuous evolution* the modified Dirac equation should read

$$H_{\mathrm{D}}\Psi(x, \rho) = -i\hbar c\left(\alpha_i \frac{\partial}{\partial x^i} + \frac{\beta}{g}\frac{\partial}{\partial \rho}\right)\Psi(x, \rho) = i\hbar\frac{\partial \Psi(x, \rho)}{\partial t}, \tag{1}$$

where g is a dimensionless constant parameter to be specified in the sequel. The variable ρ is an *intrinsic* scalar coordinate under Lorentz transformations. This coordinate is associated with the constant internal momentum gMc. Thus in Eq. (1)

we are demanding a *weak* condition upon Ψ, namely

$$\left[-c^2\nabla^2 - \left(\frac{c}{g}\right)^2 \frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial t^2}\right]\Psi(x) = 0. \tag{2}$$

For stationary states we have

$$\Psi(x) = \Psi_{E_\mathrm{D}}(x) = \Psi_0(\mathbf{p})\, e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}+gMc\rho-E_\mathrm{D}t)}. \tag{3}$$

Notice that Eq. (1) is formally equivalent to Feynman's parametrization of the Dirac equation (Aparicio *et al.*, 1995; Feynman, 1951). However, here ρ is a scalar *coordinate*, not a time *parameter*.

A direct consequence of Eq. (l) is that ρ is not a constant of the motion. In fact, in the Heisenberg picture,

$$\frac{d\rho}{dt} = \frac{i}{\hbar}\,[H_\mathrm{D}, \rho] = c\beta. \tag{4}$$

From H_D in Eq. (1) we find that $\{H_\mathrm{D}, \beta\} = 2cp_\rho$. Furthermore

$$[H_\mathrm{D}, \beta] = \{H_\mathrm{D}, \beta\} - 2\beta H_\mathrm{D} = \left(2cp_\rho H_\mathrm{D}^{-1} - 2\beta\right) H_\mathrm{D}. \tag{5}$$

Hence we get a differential equation for β,

$$\frac{d\beta}{dt} = \frac{i}{\hbar}[H_\mathrm{D}, \beta] = -\frac{2i}{\hbar}\left(\beta - cp_\rho H_\mathrm{D}^{-1}\right) H_\mathrm{D}. \tag{6}$$

Consequently, by defining

$$\eta \equiv \beta - cp_\rho H_\mathrm{D}^{-1}, \tag{7}$$

we obtain to first order a linear differential equation for η,

$$\frac{d\eta}{dt} = -\frac{2i}{\hbar}\eta H_\mathrm{D}, \tag{8}$$

whose solution is (Barut and Bracken, 1981)

$$\eta(t) = \eta(0)\, e^{-\frac{2i}{\hbar}H_\mathrm{D}t} = e^{+\frac{2i}{\hbar}H_\mathrm{D}t}\eta(0). \tag{9}$$

Therefore from (7) and (9) we find that

$$\beta(t) = cp_\rho H_\mathrm{D}^{-1} + \left(\beta - cp_\rho H_\mathrm{D}^{-1}\right)\, e^{-\frac{2i}{\hbar}H_\mathrm{D}t}, \tag{10}$$

so that

$$\frac{d\rho}{dt} = \frac{i}{\hbar}[H_\mathrm{D}, \rho] = c\beta = c^2 p_\rho H_\mathrm{D}^{-1} + c\left(\beta - cp_\rho H_\mathrm{D}^{-1}\right)\, e^{-\frac{2i}{\hbar}H_\mathrm{D}t}, \tag{11}$$

which we integrate to give

$$\rho(t) = \rho(0) + c^2 p_\rho H_\mathrm{D}^{-1}t + \frac{i\hbar c}{2}\left(\beta - cp_\rho H_\mathrm{D}^{-1}\right) H_\mathrm{D}^{-1}\, e^{-\frac{2i}{\hbar}H_\mathrm{D}t}, \tag{12}$$

with $\rho(0)$ a constant (operator) of integration. The second term of Eq. (12) grows linearly with respect to time t by analogy to the classical result. The remaining contribution to ρ describes a microscopic, high-frequency *Zitterbewegung* that, in the rest frame system, is characterized by $\hbar/2Mc$, half the Compton wavelength of the spin 1/2 particle, and an intrinsic frequency $2Mc^2/\hbar$.

As an example of a bound state problem for Eq. (1), we shall consider the minimal coupling prescription

$$p_\rho \to p_\rho - i\left(\frac{\hbar}{\lambda^2}\right)\gamma_5\rho. \tag{13}$$

Thus Eq. (1) becomes

$$H_{\mathrm{D}}\Psi(x,\rho) = -i\hbar c\left\{\alpha_i\frac{\partial}{\partial x^i} + \beta\left[\frac{\partial}{\partial\rho} + \left(\frac{1}{\lambda^2}\right)\gamma_5\rho\right]\right\}\Psi(x,\rho) = i\hbar\frac{\partial\Psi(x,\rho)}{\partial t}, \tag{14}$$

with

$$\Psi(x,\rho) = e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x} - E_{\mathrm{D}}t)}\Phi(\rho). \tag{15}$$

Notice that like in the case of the Dirac oscillator (Moshinsky and Szczepaniak, 1989), although the minimal coupling (13) is not hermitian, after replacing it into Eq. (14), the hamiltonian remains hermitian. By replacing Eq. (15) into Eq. (14) and applying H_{D} on the left-hand side of Eq. (14), the eigenvalue problem turns out to be

$$E_{\mathrm{D}}^2\Phi(\rho) = \left\{c^2\mathbf{p}^2 + c^2\left[-\hbar^2\frac{\partial^2}{\partial\rho^2} + \left(\frac{\hbar}{\lambda^2}\right)^2\rho^2 - \left(\frac{\hbar}{\lambda}\right)^2\gamma_5\right]\right\}\Phi(\rho). \tag{16}$$

Therefore the solutions for Φ have the form

$$\Phi_+(\rho) = \begin{pmatrix}\varphi(\rho)\\ \varphi(\rho)\end{pmatrix}, \qquad \Phi_-(\rho) = \begin{pmatrix}\varphi(\rho)\\ -\varphi(\rho)\end{pmatrix}. \tag{17}$$

For the eigenfuctions Φ_n of (16) we have that

$$\varphi_n(\rho) = \left[(a^\dagger)^n/\sqrt{n!}\right]\varphi_0(\rho). \tag{18}$$

Here $a^\dagger$, a are ladder operators for the one-dimensional harmonic oscillator, and $\varphi_0(\rho) = (m\omega/\pi\hbar)^{1/4}\exp(-(m\omega/2\hbar)\rho)$. By replacing Eq. (18) into Eq. (17), the associated energy eigenvalues are then

$$E_+ = \pm\sqrt{c^2\mathbf{p}^2 + 2\hbar^2\omega^2\left(n+\frac{1}{2}\right) - \hbar^2\omega^2},$$

$$E_- = \pm\sqrt{c^2\mathbf{p}^2 + 2\hbar^2\omega^2\left(n+\frac{1}{2}\right) + \hbar^2\omega^2}, \tag{19}$$

respectively. In the rest frame system ($\mathbf{p} = \mathbf{0}$) and for positive energies $E_+, E_- \geq 0$, E_+ and E_- take the same values (twofold degeneracy) except one, namely the ground state which has the value $[E_+(\mathbf{p} = \mathbf{0})]_{n=0} = 0$. Therefore there exists a $N = 1$ hidden supersymmetry associated with this oscillator, analogous to the case of $N = 2$ supersymmetry found in the Dirac oscillator (Dixit *et al.*, 1992; Martinez *et al.*, 1991). In fact from (16) we can write

$$H_\mathrm{D}^2 = (c^2\mathbf{p}^2 + 2\hbar\omega H_\mathrm{SS}), \tag{20}$$

with

$$H_\mathrm{SS} = \{Q, Q^\dagger\}, \qquad Q^2 = (Q^\dagger)^2 = 0, \tag{21}$$

where

$$Q = \frac{1}{2}(\hbar\omega)\, a\sigma_-, \qquad Q^\dagger = \frac{1}{2}(\hbar\omega)a^\dagger\sigma_+. \tag{22}$$

Here $\sigma_\pm = \sigma_1 \pm i\sigma_2$ with σ_i the Pauli matrices.

Following Martinez *et al.* (1991), we construct generators of the su(2) Lie algebra out of the supercharges Q and $Q^\dagger$ and perform a Foldy–Wouthuysen transformation to reduce H_D to the form

$$H_\mathrm{D}' = \beta(c^2\mathbf{p}^2 + 2\hbar\omega H_\mathrm{SS})^{1/2}. \tag{23}$$

This shows the stability of the Dirac vacuum.

For the case of *discrete evolution* we can choose the time interval to be $\delta t = \tau = \hbar/Mc^2$, where M is, for instance, the mass associated with the Compton wavelength of a given spin 1/2 particle m_c, or $m_\mathrm{p} = \sqrt{\hbar_\mathrm{c}/G}$ the Planck mass. Since c and $\hbar$ are natural constants, for the rest frame system Eq. (1) tells us that ρ takes also discrete values with spacing $\delta\rho = \lambda = \hbar/Mc$. Thus we replace the corresponding partial derivatives by finite differences. To have unitary evolution (Bruce, 2001; Caldirola, 1978; Farias and Recami, 1997) we make use of the so-called symmetric derivative:

$$\begin{aligned} \frac{\partial\Psi}{\partial\rho} &\to \frac{\delta\Psi}{\delta\rho} = \frac{\Psi(\rho+\delta\rho) - \Psi(\rho-\delta\rho)}{2\delta\rho} = \frac{\hbar}{\lambda}\,\sin\left(\frac{g\lambda}{\hbar}Mc\right)\Psi(x,\rho), \\ \frac{\partial\Psi}{\partial t} &\to \frac{\delta\Psi}{\delta t} = \frac{\Psi(t+\delta t) - \Psi(t-\delta t)}{2\delta t} = \frac{\hbar}{\tau}\,\sin\left(\frac{\tau}{\hbar}E\right)\Psi(x,\rho), \end{aligned} \tag{24}$$

with $E = +\sqrt{c^2\mathbf{p}^2 + M^2c^4}$. Thus

$$H_\mathrm{D}\Psi = \left[-i\hbar c\alpha\cdot\nabla + c\beta\frac{\hbar}{g\lambda}\,\sin\left(\frac{g\lambda}{\hbar}Mc\right)\right]\Psi = E_\mathrm{D}\Psi = \frac{\hbar}{\tau}\,\sin\left(\frac{\tau}{\hbar}E\right)\Psi. \tag{25}$$

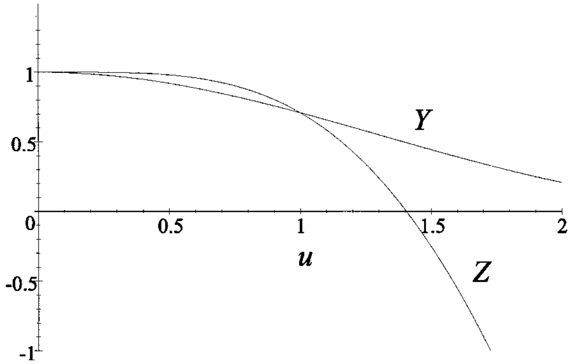

Fig. 1. The functions $Z(u) = 1 + \sin^2 u - u^2$ and $Y(u) = \sin^2 u/u^2$. They coincide only for $u = 0, 1$.

Next, from Eq. (25) we get the relation

$$E_{\mathrm{D}}^2 = E^2 - M^2c^4 + M^2c^4 \left(\frac{\sin g}{g}\right)^2 = M^2c^4 \sin^2\left(\frac{E}{Mc^2}\right), \tag{26}$$

so that

$$\left(\frac{\sin g}{g}\right)^2 = 1 + \sin^2\left(\frac{E}{Mc^2}\right) - \left(\frac{E}{Mc^2}\right)^2. \tag{27}$$

Thus in principle the only possible value for g is either 0 or 1, as shown in Fig. 1.

To have a finite solution we choose $g = 1$. Hence from Eq. (26) we find the energy eigenvalues of H_{D}:

$$E_{\mathrm{D}} = \frac{\hbar}{\tau} \sin\left(\frac{\tau}{\hbar}E\right), \tag{28}$$

for a state with eigenvalue E of H. Thus bound states have a maximum and a minimum value for the energy of the excited states: $E_{\mathrm{D}}^{+} = \pm\hbar/\tau = \pm Mc^2$. Then we can write

$$E_{\mathrm{D}} = Mc^2 \sin\left(\frac{E}{Mc^2}\right). \tag{29}$$

Hence, E_{D} reaches its maximum value for relativistic values of $E[\sim Mc^2 \lesssim (\pi/2)\, Mc^2]$, i.e., when creation and annihilation of particles take place. In such a situation we must call for a (quantum) field theory treatment.

Particularly, for the case of the Dirac oscillator (Moshinsky and Szczepaniak, 1989), with $\omega = Mc^2/\hbar$, and for s states with $E \geq 0$, we get

$$(E_{\mathrm{D}})_n = \hbar\omega \sin\left[\sqrt{2(n+1)}\right], \qquad n \in \mathbb{Z}_{+}. \tag{30}$$

Therefore we obtain an energy-bounded oscillator, $-\hbar\omega \leq (E_{\mathrm{D}})_n \leq \hbar\omega$, i.e., there exist solutions with negative energy eigenvalues, with unequal spacing between energy levels, even for positive eigenstates of H.

To conclude, for the case of continuous evolution, the alternative Dirac oscillator does exhibit supersymmetry and also ensures the stability of the Dirac vacuum.

On *small* intrinsic time scales (e.g., the proper time associated with Compton's wavelength in relativistic quantum mechanics) we conjecture that the system evolves by a sequence of real-time-like steps generated by the hamiltonian. The Schrödinger equation is obtained to nth order in the expansion real parameter. For both scattering and bound states with positive energy (negative energy) solutions for the standard Dirac equation, Eq. (29) predicts the existence of negative (positive) energies.

Finally, we want to briefly comment on the early paper by Snyder (1947; Yang, 1965). He showed that Lorentz invariance *doesnot* require that a four-dimensional spacetime be continuous. To this end, he defines a Lorentz Lie algebra and a generalized spacetime operators on homogeneous coordinates of a four-dimensional De Sitter space, together with suitable energy-momentum operators. It is clear that our approach is more straightforward: As shown in Bruce (2001), for the canonical pairs x^i, p^j, and p^i, p^j, similar commutations relations can be found for a basic relativistic quantum system, where the energy is given by Eq. (29).

We should mention that, as far as a discrete parameter has been used for time evolution, our theory lacks of explicit Lorentz invariance.

ACKNOWLEDGMENT

This work was concluded on a sabbatical semester. It received support from Dirección de Investigación, Universidad de Concepción, Chile, through Grant # 201.011.0341.0.

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